CIRCULANTS (Extract)

Alun Wyn-jones

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Preface

The work in this book grew out of a simple discovery which led to other, less simple, discoveries, and eventually to quite complicated discoveries which were all in the service of three assaults (which all failed) on Fermat's Last Theorem. Eventually, I made the really simple discovery that much was already known about the subject of my research, and appeared in the literature under the heading of "circulants" and not " cyclic matrices" as I (and others) had been calling them. I decided that a remedy to working in isolation was to publish what I knew of the subject, and to thereby familiarize the others with "circulants." That effort led to the present book.

The book assumes the reader has an understanding of undergraduate algebra as taught at an American university: linear algebra, introductory group theory, ring theory, and simple Galois Theory. Only Chapter 9 requires some calculus.

The book has been around in various forms for a long time, at least ten years. Shortly before writing the first version, I discovered the formula for the determinantal coefficients of the general circulant. This was (to me) a major accomplishment since there had been attempts since the 19^{th} century to discover such a closed formula which continued up until the early part of the twentieth century. For this reason, the book ends with that formula (Chapter 10), and its immediate consequences (Chapter 11).

Since then, I made some more discoveries. One was already known and published by Hyman Bass, but is independently derived in Chapter 7. This result is the characterization of a group of finite index in the group of units of circulant rings of prime order over the integers. In algebraic terms, the circulants are group rings whose groups are cyclic, and this is how the result was first published by Bass. The latter part of the chapter contains a generalization of this result to prime-power orders which I believe gives a better picture of the units of integral cyclic group rings.

The earlier chapters serve two purposes: to prepare for Chapter 7 which requires quite a bit of ring and cyclotomic theory, and to bring circulants into a ring-theoretic setting. This latter aim has led to much new notation which I hope meets readers' approval.

Chapter 1 introduces the circulants and their most basic properties. It ends with two formulæ for the determinant, the product-of-the-eigenvalues formula, and the rather surprising Resultant Formula.

Chapter 2 considers circulants in their traditional context of a sub-algebra of the matrices. It considers their centralizer and normalizer within the matrices. The chapter calculates the groups of circulant automorphisms over the complex and real numbers.

Chapter 3 is the first to adopt an abstract algebraic view of circulants. The chapter begins by proving a structure theorem which expresses the ring of circulants of order n in terms of the circulants having orders dividing n. The chapter then defines a host of maps between, to, and from circulants, and attempts to impose some order on these maps.

Chapter 4 introduces the "supercirculants." The professional mathematician might recognize them as an inverse limit system in the algebras of circulants. This chapter is probably the prettiest of the book containing some nice simplifications of the plethora of maps in Chapter 3.

Chapter 5 considers two sub-algebras of the circulants which are important in investigations of the circulant determinant. The first of these, the residue class circulants, led to a formula in the theory of partitions which might be new.

Chapter 6 vies with Chapter 4 for prettiness. It investigates possible tensor products over the circulants.

Chapter 7 aims to characterize the unit group of the integer circulants. This succeeds up to a finite index of the group for circulants of prime power order. This is by far the lengthiest and most technical chapter in the book.

Chapter 8 continues the study of the integer circulants from Chapter 7 but from a very different viewpoint. The chapter has a strong ring-theoretic flavor. A circulant norm is defined whose properties are shown to be typical of algebraic norms. The norm is used to find the prime elements of the integer circulants.

Chapter 9 serves as an intermission amidst algebraic abstractions. It discusses a possible application of circulants to physics and statistical mechanics.

Chapter 10 discusses an application to homogenous diophantine equations most particularly, FLT and attempts by Wendt and Sophie Germain at its resolution.

Chapter 11 contains the complete derivation of the formula for the determinantal coefficient. The book ends with an unexpected corollary of this formula which led to a simple conjecture which occupied much of my time in an attempt to prove it until I discovered a theorem of Erdös, Ginzburg, and Ziv, now called the EGZ Theorem which is equivalent to my conjecture. This theorem is proved in full.

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Preface to the Second Edition

I have made minor corrections, and added a small section in Chapter 2 on the projection operator onto the circulants from the set of general matrices.

The biggest changes are the inclusion of a theorem on arithmetical partitions in Chapter 5, the inclusion of the full proof of the formula for the determinantal coefficient in Chapter 11, and the inclusion of a new chapter, Chapter 12, on the case of circulants over finite fields. The material for the new chapter was previously a separate article.

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